KMA315 Analysis 3A: Problems 2

Solutions to the problems designated by ★ should be submitted by 11:00am on Wed the 23rd of March 2016.

1. Give and justify at least one example for each of the following:

(i) ★ a sequence $(y_n)_{n=0}^{\infty}$ of real numbers such that $\lim_{n\to\infty} y_n$ does not exist while $\lim_{n\to\infty} |y_n|$ does exist; *(2 marks)*

(ii) a sequence of real numbers that diverges but has at least one convergent subsequence; and

(iii) ★ a sequence of rational numbers that converges to an irrational number *(you may search the internet to find an example, though cite where you found it and make sure you understand the justification/explanation that you give)*, also using your example explain whether the rational numbers are a complete metric space. *(3 marks)*

2. ★ Let $(y_n)_{n=0}^{\infty}$ be the sequence of real numbers defined by $y_0 = 1$ and $y_{n+1} = \sqrt{3y_n}$ for all $n \in \mathbb{N}$. Show that:

(i) $1 \leq y_n \leq 3$ for all $n \in \mathbb{N}$; *(3 marks)*

(ii) $(y_n)_{n=0}^{\infty}$ is monotonically increasing; *(3 marks)*

(iii) $(y_n)_{n=0}^{\infty}$ converges, and furthermore find the limit $\lim_{n\to\infty} y_n$. *(3 marks)*

3. ★ Prove that if $(a_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence of real numbers and $x \in \mathbb{R}$ is a cluster point of $(a_n)_{n=0}^{\infty}$ then $\lim_{n\to\infty} a_n = x$. *(3 marks)*

4. Establish whether the following sets are: (i) open; (ii) closed; and (iii) compact:

(***Note:*** *a subset $A \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*)

(i) ★ $(0, 1] = \{r \in \mathbb{R} : 0 < r \leq 1\}$; *(1 mark)*

(ii) $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$;

(iii) ★ $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$; *(1 mark)*

(iv) $\varnothing$ (the empty set);

(v) ★ $\mathbb{R}$; *(1 mark)*

(vi) the Cantor set (use the internet to work out what that is).

There is another question over the page...

5. Give <u>and justify</u> at least one example for each of the following:

(i) ★ a sequence $(A_n)_{n=0}^{\infty}$ of open subsets of $\mathbb{R}$ whose intersection $\bigcap_{n=0}^{\infty} A_n$ is <u>not</u> open; *(3 marks)*

(ii) a subset $A \subseteq \mathbb{R}$ such that A is a proper subset of the closure of A, ie. $A \subset \overline{A}$;

(iii) ★ subsets $A \subseteq B \subseteq \mathbb{R}$ such that A is <u>not</u> compact while B is compact; *(1 mark)*

(iv) ★ a sequence $(I_n)_{n=0}^{\infty}$ of nested closed intervals of $\mathbb{R}$ such that the intersection $\bigcap_{n=0}^{\infty} I_n$ is empty. Explain why your example does not contradict the Nested Interval Property. *(3 marks)*